TRƯỜNG ĐẠI HỌC BÁCH KHOA HÀ NỘI
HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY

VIỆN TOÁN ỨNG DỤNG VÀ TIN HỌC
SCHOOL OF APPLIED MATHEMATICS AND INFORMATICS

Seminar II:

Ứng dụng toán tử tuyến tính đa trị trong việc tìm bán kính điều khiển có cấu trúc của hệ chịu đa nhiễu

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Tháng 10, 2022

ONE LOVE. ONE FUTURE

◇ Hệ điều khiển tuyến tính:

$$\dot{x} = Ax + Bu, x \in \mathbb{K}^n, A \in \mathbb{K}^{n\times n}, B \in \mathbb{K}^{n\times m}, \tag{1}$$

◇ Hệ là điều khiển được

$$\operatorname{rank}\,[A \mid B] = n, \tag{2}$$

Trong đó

$$[A \mid B] = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}. \tag{3}$$

◇ Chịu nhiễu

$$[A, B] \rightsquigarrow \left[\tilde{A}, \tilde{B}\right] = [A, B] + [\Delta_1, \Delta_2]\,. \tag{4}$$

◇ Khoảng cách đến trạng thái không điều khiển được:

$$\begin{aligned} r_{\mathbb{K}}(A, B) = \inf\{\|\Delta_1, \Delta_2\| : [\Delta_1, \Delta_2] \in \mathbb{K}^{n\times(n+m)}, \\ [A, B] + [\Delta_1, \Delta_2] \text{ là không điều khiển được}\}. \end{aligned} \tag{5}$$

Mục lục

- Ánh xạ đa trị $\mathscr{F}$

$$\mathscr{F} : \mathbb{K}^n \rightrightarrows \mathbb{K}^m \tag{6}$$

- Đồ thị của $\mathscr{F}$

$$\operatorname{gr} \mathscr{F} = \{(x, y) \in \mathbb{K}^n \times \mathbb{K}^m \colon y \in \mathscr{F}(x)\} \tag{7}$$

◇ Miền xác định của $\mathcal{F}$

$$\operatorname{dom}\mathcal{F} = \{x \in \mathbb{K}^n : \mathcal{F}(x) \neq \emptyset\} \tag{8}$$

◇ Nhân của $\mathcal{F}$

$$\ker\mathcal{F} = \{x \in \operatorname{dom}\mathcal{F} : 0 \in \mathcal{F}(x)\} \tag{9}$$

◇ Chuẩn của $\mathcal{F}$

$$\|\mathcal{F}\| = \sup\left\{\inf_{y\in\mathcal{F}(x)} \|y\| : x \in \operatorname{dom}\mathcal{F}, \|x\| = 1\right\} \tag{10}$$

Nhiễu có cấu trúc

$$[A, B] \rightsquigarrow \left[\widetilde{A}, \widetilde{B}\right] = [A, B] + D\Delta E \tag{11}$$

Bán kính điều khiển

$$r_{\mathbb{K}}^{D,E}(A, B) = \inf\left\{\|\Delta\| : \Delta \in \mathbb{K}^{l\times q}, \left[\widetilde{A}, \widetilde{B}\right] \text{ không điều khiển được}\right\} \tag{12}$$

◇ Ánh xạ đơn trị

$$\begin{aligned} W_\lambda &: \mathbb{K}^{m+n} \to \mathbb{K}^n \\ W_\lambda(z) &= \begin{bmatrix} A - \lambda I & B \end{bmatrix} z, \quad \lambda \in \mathbb{C}. \end{aligned} \tag{13}$$

◇ Ánh xạ đa trị

$$\begin{aligned} EW_\lambda^{-1}D &: \mathbb{K}^l \rightrightarrows \mathbb{K}^q \\ \left(EW_\lambda^{-1}D\right)(u) &= E\left(W_\lambda^{-1}(Du)\right). \end{aligned} \tag{14}$$

Nếu $\mathbb{K} = \mathbb{C}$ thì:

$$r_{\mathbb{C}}^{D,E}(A, B) = \frac{1}{\sup_{\lambda \in \mathbb{C}} \left\| E W_{\lambda}^{-1} D \right\|}. \tag{15}$$

- Cho $G \in \mathbb{C}^{n\times p}$, $\mathrm{rank}_{\mathsf{row}}(G) = n$, $\mathcal{F}_G(z) = Gz$

$$d\left(0, \mathcal{F}_G^{-1}(y)\right) = \left\|G^{\dagger}y\right\|, \quad \left\|\mathcal{F}_G^{-1}\right\| = \left\|G^{\dagger}\right\|. \tag{16}$$

- Cho $G \in \mathbb{C}^{n\times p}$, $\mathcal{F}_G(z) = Gz$

$$\mathcal{F}_G^{\dagger}(y) = \begin{cases} z \quad s.t. \quad Gz = y, \|z\| = d\left(0, \mathcal{F}_G^{-1}(y)\right) & y \in \operatorname{im} \mathcal{F}_G, \\ \emptyset & y \notin \operatorname{im} \mathcal{F}_G. \end{cases} \tag{17}$$

Mục lục

$$\dot{x} = Ax(t) + Bu(t), \tag{18}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{19}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{20}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \delta_1 & \delta_1 + 1 & \delta_2 + 1 \\ \delta_1 + 1 & \delta_1 & \delta_2 \end{bmatrix}. \tag{21}$$

$$[A, B] \rightsquigarrow [A, B] + D\Delta E, \tag{22}$$

$$D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{23}$$

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

$$\begin{aligned}\left(E\left[A-\lambda I, B\right]^{-1} D\right)(v) &= E\left[A-\lambda I, B\right]^{-1}\begin{bmatrix}1\\1\end{bmatrix} v\\ &= E\left[A-\lambda I, B\right]^{-1}\begin{bmatrix}v\\v\end{bmatrix} \qquad (25)\end{aligned}$$

$$\begin{aligned}
[A-\lambda I, B]^{-1}\begin{bmatrix} v \\ v \end{bmatrix} &= \left\{ \begin{bmatrix} p \\ q \\ r \end{bmatrix} : [A-\lambda I, B]\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} v \\ v \end{bmatrix}, \forall p, q, r \in \mathbb{C} \right\} \\
&= \left\{ \begin{bmatrix} p \\ q \\ r \end{bmatrix} : \begin{bmatrix} q+r-\lambda p \\ q-p\lambda \end{bmatrix} = \begin{bmatrix} v \\ v \end{bmatrix}, \forall p, q, r \in \mathbb{C} \right\}. \qquad (26)
\end{aligned}$$

$$
\begin{aligned}
\left(E\left[A-\lambda I, B\right]^{-1} D\right)(v) &= \left\{ E \begin{bmatrix} p \\ q \\ r \end{bmatrix} : \begin{bmatrix} q+r-\lambda p \\ q-p\lambda \end{bmatrix} = \begin{bmatrix} v \\ v \end{bmatrix}, \forall p, q, r \in \mathbb{C} \right\} \\
&= \left\{ \begin{bmatrix} p+q \\ r \end{bmatrix} : \begin{bmatrix} q+r-\lambda p \\ q-p\lambda \end{bmatrix} = \begin{bmatrix} v \\ v \end{bmatrix}, \forall p, q, r \in \mathbb{C} \right\} \\
&= \left\{ \begin{bmatrix} q+v+q\lambda \\ v-q+\lambda\left(v+q\lambda\right) \end{bmatrix} : q \in \mathbb{C} \right\}. \qquad (27)
\end{aligned}
$$

$$ax_1 + x_2 = b \tag{28}$$

$$x_1 = q + v + q\lambda, \tag{29}$$

$$x_2 = v - q + \lambda\left(v + q\lambda\right). \tag{30}$$

- $q = q_0,\ q = q_1 \implies a = 1 - \lambda,\ b = 2v.$

$$\left(1 - \lambda\right)x_1 + x_2 = 2v, \tag{31}$$

$$x_1 = q + v + q\lambda, \tag{32}$$

$$x_2 = v - q + \lambda\left(v + q\lambda\right). \tag{33}$$

◇ $\lambda = -1$

$$x_1 = v, \tag{34}$$
$$x_2 = 0, \tag{35}$$
$$d\left(0, \left(E\left[A - \lambda I, B\right]^{-1} D\right)(v)\right) = |v|. \tag{36}$$

◇ $\lambda \neq -1$, $(\mathbb{C}, \|\cdot\|_\infty)$

$$\begin{aligned} 2|v| &= |(1-\lambda)x_1 + x_2| \\ &\leq |(1-\lambda)x_1| + |x_2| \\ &\leq (|1-\lambda| + 1)\max\{|x_1|, |x_2|\} \\ &= (|1-\lambda| + 1)\left\|\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\|_\infty \end{aligned} \tag{37}$$

◇ $\lambda \neq -1$, $(\mathbb{C}, \|\cdot\|_\infty)$

$$\frac{2|v|}{|\lambda - 1| + 1} \leq \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_\infty . \tag{38}$$

◇ Dấu bằng:

$$x_1 = e^{i\varphi} x_2, \tag{39}$$

$$x_2 = \frac{2v}{1 + |\lambda - 1|} \tag{40}$$

$$-(\lambda - 1)\, e^{i\varphi} = |\lambda - 1| \tag{41}$$

$$
\begin{aligned}
\left\| E\left[A-\lambda I, B\right]^{-1} D \right\| &= \sup_{|v|=1} d\left(0, E\left[A-\lambda I, B\right]^{-1} D\left(v\right)\right) \\
&= \begin{cases} \sup_{|v|=1}\left\{ \inf \left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_\infty \right\} & \lambda \neq -1, \\ \sup_{|v|=1}\left\{ \inf |v| \right\} & \lambda = -1, \end{cases} \\
&= \begin{cases} \sup_{|v|=1}\left\{ \frac{2|v|}{|\lambda-1|+1} \right\} & \lambda \neq -1, \\ \sup_{|v|=1}\left\{ |v| \right\} & \lambda = -1, \end{cases} \\
&= \begin{cases} \frac{2}{|\lambda-1|+1}, & \lambda \neq -1, \\ 1 & \lambda = -1. \end{cases}
\end{aligned} \tag{42}
$$

$$\sup_{\lambda\in\mathbb{C}} \left\| E\left[A-\lambda I, B\right]^{-1} D \right\| = 2, \tag{43}$$

$$r_{\mathbb{C}}^{D,E}(A, B) = \frac{1}{2}. \tag{44}$$

◇ Hệ chịu đa nhiễu

$$[A, B] \rightsquigarrow \left[\tilde{A}, \tilde{B}\right] = [A, B] + \sum_{i=1}^{N} D_i \Delta_i E_i. \tag{45}$$

◇ Nhiễu

$$\Delta = (\Delta_1, \dots, \Delta_N) \in \Pi_{i=1}^{N} \mathbb{C}^{l_i \times q_i} \tag{46}$$

$$\|\Delta\| = \sum_{i=1}^{N} \|\Delta_i\| \tag{47}$$

◇ Bán kính điều khiển

$$r_{\mathbb{C}}^{\mathsf{mp}}(A,B) = \inf\left\{\|\Delta\| : \Delta = (\Delta_i)_{i\in\underline{N}}, [A,B] + \sum_{i=1}^{N} D_i\Delta_i E_i \text{ không điều khiển được}\right\}. \tag{48}$$

◇ Ký hiệu

$$P \preceq Q \iff \|Px\| \le \|Qx\|, \quad \forall x \in \mathbb{C}^m \tag{49}$$

◇ $H \in \mathbb{C}^{k\times(n+m)}$, $E_i \preceq H \, \forall i \in \underline{N}$

$$\left[\max_{i\in\underline{N}} \sup_{\lambda\in\mathbb{C}} \left\|HW_\lambda^{-1}D_i\right\|\right]^{-1} \le r_{\mathbb{C}}^{\mathsf{mp}}(A,B) \le \left[\max_{i\in\underline{N}} \sup_{\lambda\in\mathbb{C}} \left\|E_iW_\lambda^{-1}D_i\right\|\right]^{-1}. \tag{50}$$

◇ $E_i = \alpha_i E_1$, $\alpha_i \in \mathbb{C} \, \forall i \in \underline{N}$,

$$r_{\mathbb{C}}^{\mathsf{mp}}(A,B) = \left[\max_{i\in\underline{N}} \sup_{\lambda\in\mathbb{C}} \left\|E_iW_\lambda^{-1}D_i\right\|\right]^{-1}. \tag{51}$$

$$\dot{x} = Ax(t) + Bu(t), \tag{52}$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \tag{53}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{54}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \delta_1 & 1+\delta_2 & 0 \\ -2+\delta_3 & 0 & 1+\delta_4 \end{bmatrix}. \tag{55}$$

$$D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{56}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{57}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{58}$$

$$
\begin{aligned}
E_1\left[A-\lambda I, B\right]^{-1} D_1 \begin{bmatrix} x \\ y \end{bmatrix} &= E_1\left[A-\lambda I, B\right]^{-1} \begin{bmatrix} x \\ 0 \end{bmatrix} \\
&= \left\{ \begin{bmatrix} p \\ x+\lambda p \\ 0 \end{bmatrix} : p \in \mathbb{C} \right\}, \qquad (59) \\
E_2\left[A-\lambda I, B\right]^{-1} D_2 \begin{bmatrix} x \\ y \end{bmatrix} &= E_2\left[A-\lambda I, B\right]^{-1} \begin{bmatrix} 0 \\ y \end{bmatrix} \\
&= \left\{ \begin{bmatrix} p \\ 0 \\ y+\left(\lambda^2+2\right) p \end{bmatrix} : p \in \mathbb{C} \right\}. \qquad (60)
\end{aligned}
$$

$$\left\| E_1 \left[A - \lambda I, B \right]^{-1} D_1 \right\| = \frac{1}{1 + |\lambda|}, \tag{61}$$

$$\left\| E_2 \left[A - \lambda I, B \right]^{-1} D_2 \right\| = \frac{1}{1 + |\lambda^2 + 2|}. \tag{62}$$

$$H\left[A-\lambda I,B\right]^{-1}D_1\begin{bmatrix} x\\ y\end{bmatrix}=\left[A-\lambda I,B\right]^{-1}\begin{bmatrix} x\\ 0\end{bmatrix}$$
$$=\left\{\begin{bmatrix} p\\ x+\lambda p\\ \lambda x+\left(\lambda^2+2\right)p\end{bmatrix}:p\in\mathbb{C}\right\},\tag{63}$$
$$H\left[A-\lambda I,B\right]^{-1}D_2\begin{bmatrix} x\\ y\end{bmatrix}=\left[A-\lambda I,B\right]^{-1}\begin{bmatrix} 0\\ y\end{bmatrix}$$
$$=\left\{\begin{bmatrix} p\\ \lambda p\\ y+\left(\lambda^2+2\right)p\end{bmatrix}:p\in\mathbb{C}\right\}.\tag{64}$$

◇ Chọn $p = 0$ và $p = -\frac{x}{\lambda}$, $\forall\lambda \in \mathbb{C}$:

$$d\left(0, H\left[A - \lambda I, B\right]^{-1} D_1 \begin{bmatrix} x \\ y \end{bmatrix}\right) \leq \min\left\{\max\left\{|x|, |\lambda x|\right\}, \left|\frac{2x}{\lambda}\right|\right\} \leq \sqrt{2}\,|x|. \tag{65}$$

$$\Longrightarrow \left\|H\left[A - \lambda I, B\right]^{-1} D_1\right\| \leq \sqrt{2} \quad \forall\lambda \in \mathbb{C}. \tag{66}$$

◇ Dấu bằng với $\lambda = \sqrt{2}i$:

$$\sup_{\lambda\in\mathbb{C}} \left\|H\left[A - \lambda I, B\right]^{-1} D_1\right\| = \sqrt{2}. \tag{67}$$

◇ Tương tự, với $p = 0$:

$$d\left(0, H\left[A - \lambda I, B\right]^{-1} D_2 \begin{bmatrix} x \\ y \end{bmatrix}\right) \leq |y|, \tag{68}$$

$$\left\| H\left[A - \lambda I, B\right]^{-1} D_2 \begin{bmatrix} x \\ y \end{bmatrix} \right\| \leq 1 \quad \forall \lambda \in \mathbb{C}. \tag{69}$$

◇ Dấu bằng khi $\lambda = \sqrt{2}i$:

$$\sup_{\lambda \in \mathbb{C}} \left\| H\left[A - \lambda I, B\right]^{-1} D_2 \right\| = 1. \tag{70}$$

- Ước lượng bán kính điều khiển

$$\frac{1}{\sqrt{2}} = \frac{1}{\max\left\{1, \sqrt{2}\right\}} \leq r_{\mathbb{C}}^{\mathsf{mp}}(A, B) \leq \frac{1}{\max\left\{\sup \frac{1}{1+|\lambda|}, \sup \frac{1}{1+|\lambda^2+2|}\right\}} = 1. \quad (71)$$

Cảm ơn mọi người đã chú ý lắng nghe!