Inner Products and Cosines

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Given two vectors $\vec{u}, \vec{v} \in \mathcal{R}^n$, the inner product or dot product is defined as

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i \tag{1}$$

where u_i, v_i are components of $\vec{u}$ and $\vec{v}$ relative to some orthonormal basis of $\mathcal{R}^n$. Often taken for granted is the relationship

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta \tag{2}$$

where θ is the angle between the vectors $\vec{u}$ and $\vec{v}$. But how to show this is true? And what does θ mean in higher dimensions?

Well, first of all, any two vectors in a finite dimensional space determine a plane. That's because any three points determine a plane and two vector endpoints, together with their common origin, determine these three points. So it suffices to show that (1) and (2) are equivalent in two dimensions.

Rotate the vectors $\vec{u}$ and $\vec{v}$ so that $\vec{u}$ lies along the positive x-axis. In this position we have $\vec{u} = (u_x, 0)$ and $\vec{v} = (v_x, v_y)$. Their inner product is

$$\begin{aligned} \langle \vec{u}, \vec{v} \rangle &= \langle (u_x, 0), (v_x, v_y) \rangle \qquad (3) \\ &= u_x v_x + 0 v_y \\ &= ||\vec{u}||||\vec{v}|| \cos\theta \end{aligned}$$

We have $u_x = ||\vec{u}||$ since $\vec{u}$ lies along the x-axis and $v_x = ||\vec{v}|| \cos\theta$ since θ is the angle $\vec{v}$ makes with the x-axis.

But generally one can't depend on one of the vectors lying along the x-axis. However, the inner product is invariant under rotations. A rotation by an angle ϕ in two dimensions is represented by

$$R = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

Rotation of $\vec{u}$ by angle ϕ is

$$Ru = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} u_x\cos\phi + u_y\sin\phi \\ -u_x\sin\phi + u_y\cos\phi \end{bmatrix}$$

For any two vectors $\vec{u} = (u_x, u_y)$ and $\vec{v} = (v_x, v_y)$, the impact of a rotation on their inner product is

$$\begin{aligned}
\langle (Ru)^T, Rv\rangle &= \begin{bmatrix} u_x\cos\phi + u_y\sin\phi \\ -u_x\sin\phi + u_y\cos\phi \end{bmatrix}^T \cdot \begin{bmatrix} v_x\cos\phi + v_y\sin\phi \\ -v_x\sin\phi + v_y\cos\phi \end{bmatrix} \\
&= (u_x\cos\phi + u_y\sin\phi)(v_x\cos\phi + v_y\sin\phi) \\
&\quad +(-u_x\sin\phi + u_y\cos\phi)(-v_x\sin\phi + v_y\cos\phi) \\
&= u_xv_x\cos^2\phi + (u_yv_x + u_xv_y)\cos\phi\sin\phi + u_yv_y\sin^2\phi \\
&\quad +u_xv_x\sin^2\phi - (u_yv_x + u_xv_y)\cos\phi\sin\phi + u_yv_y\cos^2\phi \\
&= u_xv_x(\cos^2\phi + \sin^2\phi) + u_yv_y(\cos^2\phi + \sin^2\phi) \\
&= u_xv_x + u_yv_y \\
&= \langle u, v\rangle
\end{aligned}$$

So the formula is still true after any rotation.